Non-normal distribution assumptions: skew and kurtosis

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Frontier Advisors

About us

Frontier has been at the forefront of institutional investment advice in Australia for over 25 years and provides advice on over $600 billion of assets across the superannuation, charity, public sector, insurance and university sectors.

Frontier's purpose is to empower our clients to advance prosperity for their beneficiaries through knowledge sharing, customisation, technology solutions and an alignment and focus unconstrained by product or manager conflict.

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Vivian Xu joined Frontier Advisors in 2022 in the Capital Markets and Asset Allocation Team. Her role focuses on quantitative analysis and macro economic analysis. Vivian is pursuing a PhD in Finance from the University of Melbourne. Her research interest covers mutual fund management, corporate finance and emerging markets. Vivian has published articles in top ranking peer-reviewed journals.

Vivian holds a Bachelor degree in Economics from the Renmin University of China and a Master in Finance from the University of Melbourne. Vivian is CFA level two qualified.

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Quantitative Developer

Huan joined Frontier Advisors as a Quantitative Developer in March 2021. Huan is responsible for research and development of quantitative solutions within the Technology team. Prior to joining Frontier, Huan was a Research Analyst in Kaiser Trading Group (KTG) where he was focused on developing Automatic Execution Systems and intra-day trading strategies. Before KTG, Huan worked as a research academic at The University of Melbourne and Deakin University after he obtained his PhD in System Engineering from The Australian National University.

Huan holds a B.Sc. and M.Sc. degree from Northeastern University, China, majoring in Applied Mathematics.

INTRODUCTION

Portfolio modelling implications of skew and kurtosis

The traditional approach in modelling portfolio outcomes has been to use a mean-variance approach which assumes all asset returns are normally distributed. However, we know from history that asset returns are seldom 'normal'.

Frontier has built a stochastic model in *Portfolio Analytics* that simulates return distributions. This model allows for non-normal assumptions, and thus increased analytics on the path of outcomes and results at extremes. This paper analyses historical asset class return distributions, including skew and kurtosis. We then use non-normal distributions to model outcomes to show the potential impact on portfolios.

Background

Normal versus non-normal assumptions

The traditional approach in modelling portfolio outcomes has been to use a mean-variance approach.

- Benefit: intuitive in understanding modelling results.
- Drawback: involves the assumption that all asset classes are ‘normally’ distributed.

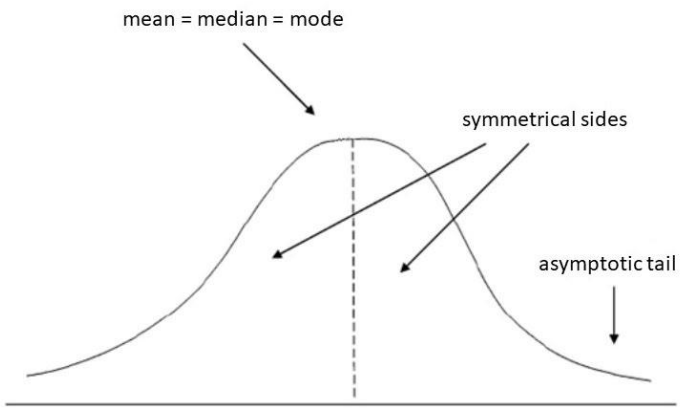
However, we know from history that asset class returns are seldom ‘normal’.

- In particular, asset class returns usually exhibit what is referred to as ‘skew’ and ‘kurtosis’.

Skewness is a measure of asymmetry of the probability distribution of returns about its mean.

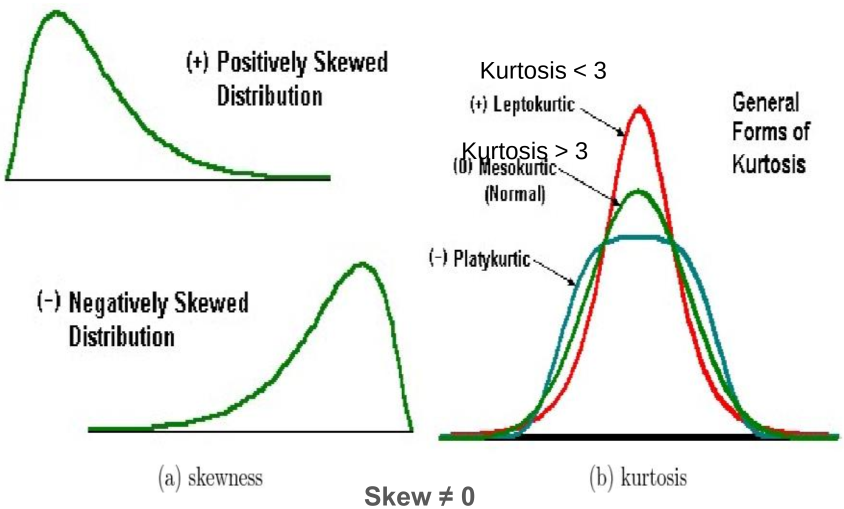
Kurtosis is a measure used to describe the degree to which scores cluster in the tail or the peak of a frequency distribution.

Normally distributed asset class returns



Skew = 0
Kurtosis = 3

Non-normally distributed asset class returns



Skew ≠ 0
Kurtosis ≠ 3

Modelling approach

Normal versus non-normal assumptions

Show how portfolio modelling results change with the skew and kurtosis assumptions.

Step 1

Calculate historical return skew and kurtosis for each asset class.

Step 2

Use the outputs from the historical values to replace the default inputs for portfolio modelling.

Step 3

Using Monte Carlo simulation to generate simulation paths based on normal versus non-normal assumptions.

Step 4

Compare the portfolio modelling results from default assumptions and historical assumptions.

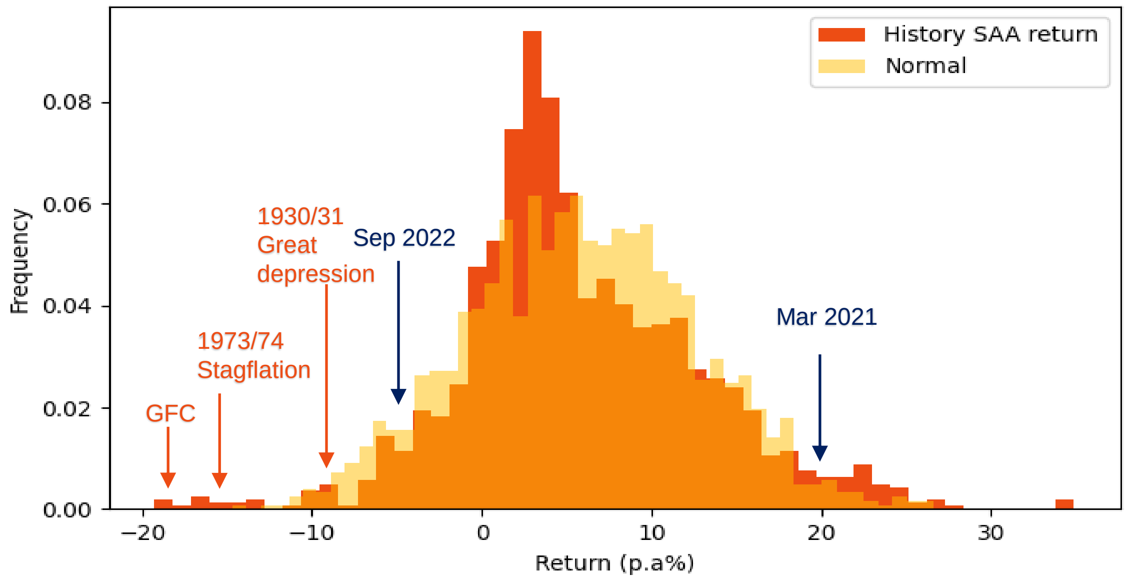
Step 5

Check result consistency across various portfolios and model inputs.

History portfolio return distribution

Historical distribution of a balanced portfolio looks different to normal distribution

Chart 1: Historical SAA distribution



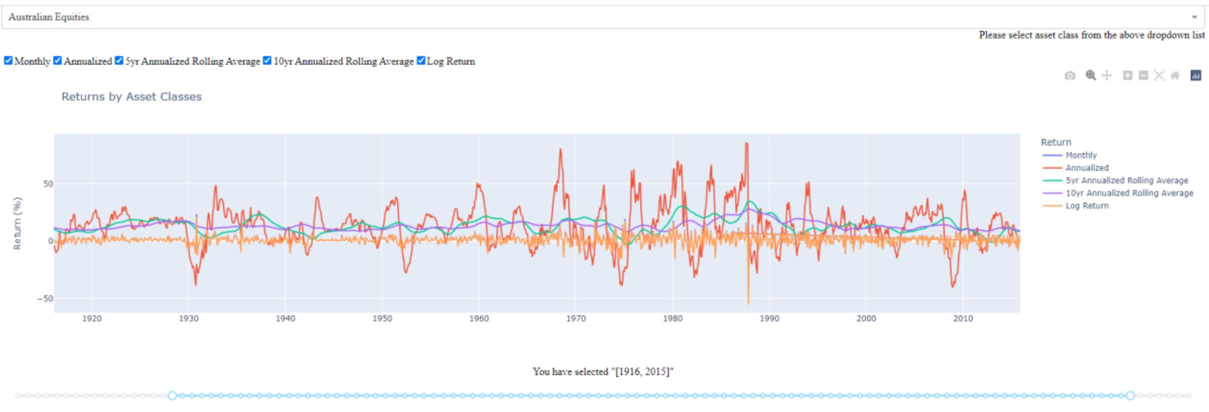
Correlation and diversification: The correlation between different asset classes is a crucial factor in portfolio construction. Ideally, a balanced portfolio includes assets with low or negative correlations, which means they do not move in perfect sync with each other. Diversification helps reduce risk by spreading investments across different asset classes. The presence of correlations and diversification can affect the shape of the historical distribution, potentially leading to non-normal patterns.

Tail events and extreme returns: Normal distributions assume that extreme events are rare and that returns are symmetrically distributed around the mean. However, financial markets are known to experience occasional tail events, such as market crashes or unexpected booms, which can result in significant deviations from normality. These extreme returns can introduce fat tails and make the historical distribution of a balanced portfolio appear different from a normal distribution.

It is important to note that while the historical distribution may differ from a normal distribution, it does not necessarily indicate a flaw in portfolio construction. In fact, a balanced portfolio aims to provide a risk-return profile that aligns with an investor's goals and preferences, rather than adhering strictly to a normal distribution. The key is to carefully consider factors such as correlation and diversification when creating a portfolio that meets one's specific investment objectives and risk tolerance.

Historical asset class return distributions

We built an internal interactive app for analysing the historical return distributions of many asset classes



Return Statistics of Selected Period

Stats	Monthly Return	Annulized Return
Mean (%)	0.9928	12.7736
Std (%)	4.2506	17.9771
Skewness*	-1.3023	0.2634
Excess Kurtosis*	14.8524	1.5199
Median (%)	1.1563	13.0485

Asset class dropdown list

Adjustable time frame

Stats output to spreadsheet for all asset classes

Histogram with normal curve

Rolling skew and kurtosis

Frontier's *Capital Market and Asset Allocation Team* are building several protype interactive quantitative models, like the one presented on this page. In the future, we envisage that some of these tools will be made available for clients through Frontier's *Partner's Platform.*

Historical asset class return distributions

Skew and kurtosis

We examined 66 asset classes and found:

- 28 have negative skew
- 47 have positive excess kurtosis (compared to normal)

Table 1 shows the historical skew and kurtosis for the main asset classes.

However, we note caution when using historical statistics.

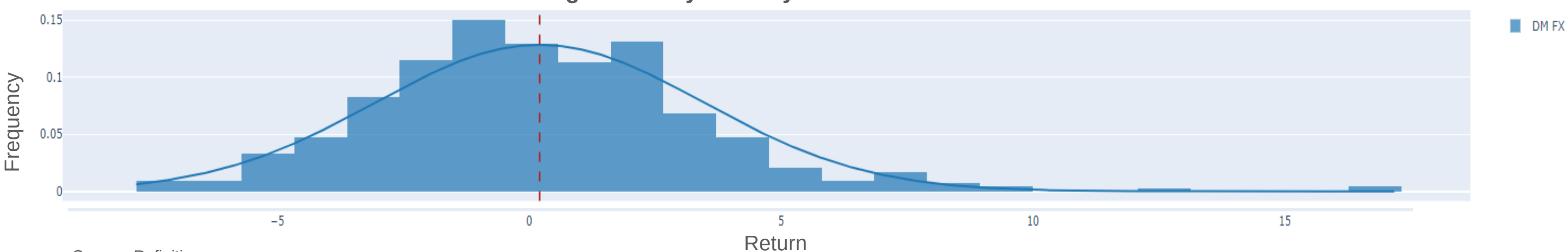
Skew and kurtosis are highly sensitive to extreme values in the data series, making them less informative for shorter sample period.

Historical distributions sometimes have multiple peaks, inconsistent with most portfolio modelling tool assumptions, as per the example on Chart 2.

Table 1: Summary of key asset class historical statistics

Asset class	Start date	Skew	Kurtosis
Australian equities	2/28/1900	0.28	4.74
International equities - DM and EM (H)	4/30/1986	-0.67	3.59
International equities emerging markets (H)	1/31/1926	0.56	3.68
International listed infrastructure (H)	4/30/1986	-0.67	3.22
Australian infrastructure	4/30/1986	0.30	4.58
Australian listed property	1/31/1980	-1.31	6.44
Australian core property	2/28/1981	-0.86	3.95
Australian sovereign bond	2/28/1900	-1.43	5.92
International sovereign bonds (H)	2/28/1900	1.98	15.51
International high yield debt (H)	4/30/1986	1.56	7.81
Cash	2/28/1900	0.90	10.04
DM foreign currency	4/30/1986	1.63	5.33

Chart 2: DM foreign currency monthly return distribution 1986-2021



Source: Refinitiv

Adjustments to historical skew and kurtosis values

We account for a variety of factors to qualitatively adjust for the historical values.

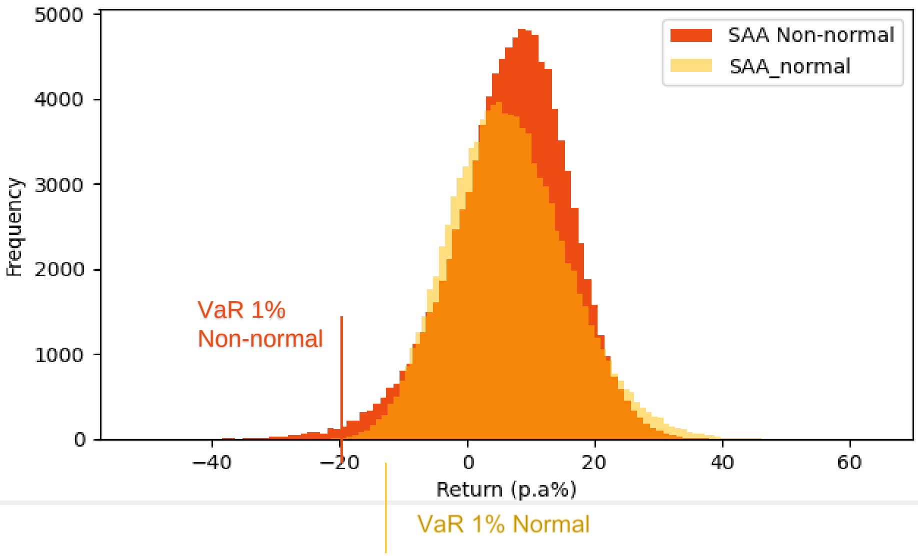
To allow for non-normal distribution assumptions in our portfolio modelling capabilities, the Capital Market and Asset Allocation Team (CMAAT) has created a default set of non-normal skew and kurtosis assumptions to be linked into future version of Portfolio Analytics Stochastic Simulator. The purpose of this is to provide Portfolio Analytics users with a base case non-normal scenario which reflects historical return distribution that also incorporates qualitative adjustments based on our experience and research.

Ensured consistency within a broad asset class	**Tried to use longer term data but also conscious of changes in regimes and relevance**	**Ensured relativities of assumptions were reasonable**	**Adjusted values are cross-checked with existing academic and industry research**	**Accounted for simplicity and technical constraints**
• Generally, we allocate the same skew and kurtosis values within a broad asset class (e.g. all equities) unless there is a strong view of having a different assumption.	• We apply long-term data (1900-2019) to produce baseline historical values. • In the case of bonds, we looked beyond the last 30 years (as bond yields have only moved in one direction), and also considered the effect of the interest rate hiking cycle and the particularly high volatility in bond yields during war periods.	• Hedged versus unhedged equity. • Equities versus bonds. • Domestic versus international.	• See appendix for the literature review.	• Rounded numbers are used both for simplicity and due to technical constraints (the PA risk simulator currently only allows for integer skew and kurtosis). • Non-normal simulation requires certain mathematical relationship between skew and kurtosis parameters for the simulation to run through.

Modelling outcome

Portfolio modelling with a particular focus on the extreme tail risk

Chart 3: Simulated return distribution normal versus non-normal



The values are after tax (15% tax rate) return with ten-year SAA allocation. Assumption used for the simulation is the latest 2023 ten-year assumption. There may be slight differences in simulated values in each run but broadly stable.

Table 2: Summary stats of simulated distribution

	SAA normal	SAA non-normal	Change
Mean	6.74%	6.78%	→
Standard deviation	9.34%	9.29%	→
50%	6.15%	7.49%	↑
75%	12.66%	13.02%	↑
99%	31.09%	26.23%	↓
Skew	0.38	-0.51	↓
Kurtosis	0.27	0.93	↑
-VaR (5%)	-7.64%	-9.54%	↓
-CVaR (5%)	-10.60%	-15.34%	↓
-VaR (1%)	-12.49%	-18.74%	↓
-CVaR (1%)	-14.79%	-24.57%	↓
Probability of negative return	24.35%	21.1%	↓
Probability of exceeding CPI + 3%	52.89%	59.07%	↑
Probability of exceeding 5%	55.09%	61.09%	↑

- Chart 3 shows non-normal distribution is more negatively skewed and has a fatter left tail compared to the normal one, suggesting a bigger loss in market drawdowns even though overall non-normal has a higher probability of exceeding the target return and lower probability of a negative return.
- VaR and CVaR quantify the potential losses in extreme scenarios.
  - By our estimation of VaR, for 5% of the time, or 0.6 month out of every 12, the portfolio is expected to lose at least 7.64% with normal distribution and 9.54% with non-normal. The difference in losses is even more salient at further end of the tail - for 1% of the time, the portfolio is expected to lose at least 12.49% and 18.74% for normal vs non-normal.
  - CVaR is an extension of VaR that gives the average amount of loss beyond the cut off point given a loss event. By our estimation of CVaR, given the same balanced fund, during the 5% (1%) time of market drawdown, the average loss beyond the cut off point could range up to 10.60% (14.79%) for normal and 15.34% (24.57%) for non-normal.

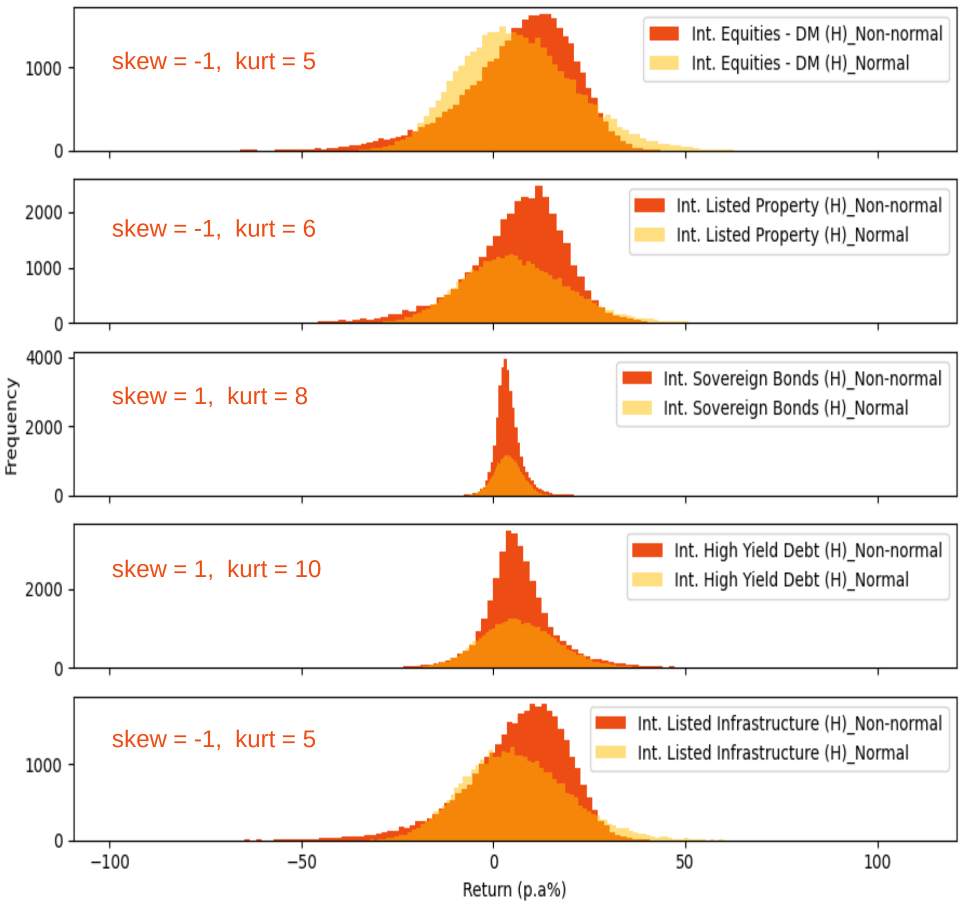
Modelling outcome

Return distribution for individual asset classes

The shape of asset distribution varies for different asset classes under non-normal assumptions, highlighting the potential impacts of considering non-normal assumptions in portfolio modelling.

- All non-normal distributions share the same return and risk with normal distributions but vary in skew and kurtosis which results in different shapes.
  - Equities, listed property and infrastructure exhibit negative skew and with an extended tail to the left, reflecting higher downside risk in market drawdowns.
  - Historical distributions of most asset classes show a fatter tail (higher kurtosis) than normal distribution.
- Including multiple asset classes in one portfolio provides *diversification* benefits because different asset classes often have different return distributions and are not perfectly correlated. This can help reduce the overall risk of the portfolio and can also help create more favorable distribution shapes (for instance, by combining assets to create a portfolio with higher positive skew or lower kurtosis).
- Less well-diversified portfolios or portfolios with high weights in asset classes that have non-normal distributions can see more significant impacts from the skewness and kurtosis of these distributions. This can lead to *more extreme portfolio outcomes* and may require more careful risk management.

Chart 4: Return distribution by asset class normal versus non-normal



Source: Frontier Advisors

Conclusion

In conclusion, this research report shows the extension of moving beyond the traditional mean-variance approach in modelling portfolio outcomes, which assumes a normal distribution of asset returns. The traditional normal mean-variance approach is a reasonable estimation of market movements. However, historical data indicates that asset returns are often non-normal, which requires a more sophisticated approach to incorporate in portfolio modeling.

Incorporating these features into risk assessments can lead to additional insights of the overall risks associated with a portfolio, enabling expanded simulations of potential portfolio performance. This is particularly pertinent in stress testing and scenario analysis, where understanding outcomes in extreme market conditions is key.

Frontier's development of a stochastic model in *Portfolio Analytics* represents a significant advancement in this regard. This model, which simulates return distributions, allows for non-normal distributions, thereby providing a more sophisticated analysis of potential outcomes and extreme results. Our paper delves into the analysis of historical asset return distributions, including skew and kurtosis, and employs non-normal distributions to model outcomes, demonstrating the potential impact on portfolios.

This research is part of our ongoing efforts to refine and enhance portfolio modelling techniques. A potential extension to the project is considering skewness and kurtosis in the context of different macroeconomic regimes, which can enhance our understanding of how a portfolio might behave under various economic conditions, thereby aiding in the management of risks associated with economic shifts.

Want to learn more?

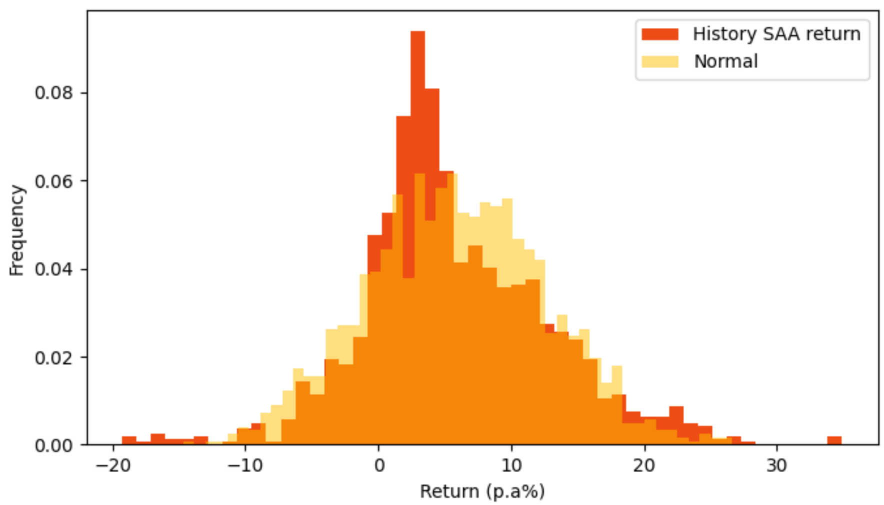
Please reach out to Frontier Advisors if you have any questions or visit frontieradvisors.com.au for more information.

Appendix

Portfolio return normal versus history

Portfolio modelling outcome for historical distribution of a balanced portfolio

Exhibit 1: Portfolio return normal vs history



Source: Frontier Advisors

The values are portfolio return with ten-year SAA allocation assuming no rebalance. Normal distribution is generated based on the historical mean and std.

Exhibit 2: Summary stats of historical vs normal distribution

	SAA normal	SAA historical	Change
Mean	5.79%	5.79%	→
Standard deviation	7.07%	7.21%	→
50%	4.56%	5.88%	↑
75%	10.26%	10.57%	↑
99%	22.45%	24.10%	↑
Skew	0.33	-0.08	↓
Kurtosis	1.06	-0.01	↓
-VaR (5%)	-4.91%	-4.27%	↑
-CVaR (5%)	-7.88%	-8.57%	↓
-VaR (1%)	-10.26%	-11.68%	↓
-CVaR (1%)	-12.11%	-15.98%	↓
Probability of negative return	18.60%	15.54%	↓
Probability of exceeding CPI + 3%	81.19%	83.50%	↑
Probability of exceeding 5%	81.19%	83.57%	↑

- Exhibit 1 shows the historical distribution is more negatively skewed and has fatter tails on both sides compared to the normal one, suggesting bigger gain/loss in market booms/drawdowns, indicating more frequent extreme deviations than would be predicted by a normal distribution. This is common in financial markets, where 'black swan' events (very rare and extreme events) can have significant impacts.
- VaR and CVaR quantify the potential losses in extreme scenarios. By our estimation of VaR, for 5% of the time, the portfolio is expected to lose at least 4.91% with normal distribution and 4.27% with historical. While normal seems to produce a smaller loss, the difference in losses at the further end of the tail inverted - for 1% of the time, the portfolio is expected to lose at least 10.26% and 11.68% for normal versus historical.
  - CVaR is an extension of VaR which gives the average amount of loss beyond the cut-off point given a loss event. By our estimation of CVaR, given the same balanced fund, during the 5% (1%) time of market drawdown, the average loss beyond the cut-off point could range up to 7.88% (12.11%) for normal and 8.57% (15.98%) for historical.

Method of generating multivariate normal and non-normal simulations

Generate multivariate normal distributions using Cholesky decomposition

X_1
X_2
X_3
.
.
X_n

- Cholesky decomposition of the target correlation matrix
- Transformation using the target return, risk, and the result of Cholesky decomposition

Y_1
Y_2
Y_3
.
.
Y_n

Generate multivariate non-normal distributions using Vale–Maurelli method

X_1
X_2
X_3
.
.
X_n

Cholesky decomposition method to generate correlated normal samples with an **intermediate** correlation matrix

V_1
V_2
V_3
.
.
V_n

Fleishman's transformation for V_i

$Y_i = a_i + b_i V_i + c_i V_i^2 + d_i V_i^3$

Y_1
Y_2
Y_3
.
.
Y_n

Standard uncorrelated normal random variables

Standard normal random variables with **intermediate** correlation matrix calculated from desired correlation matrix

Non-normal random variables with desired mean, std, skew, and kurtosis for each random variable and **target** correlation matrix

Literature review

We reviewed considerable published research to ensure the base case values were reasonable

Exhibit 3: Summary of literature on skew and kurtosis

Source	Credit Suisse	Doan, Lin	Doan, Lin, Zurbruegg		Gormsen, Jensen	Guidolin, Nicodano								Morningstar							PGIM				Racicot, Theoret		Bali, Mo, Tang	Patton	
Year of issuance	2012	2012	2009		2020	2010								2011							2021				2022		2008	2004	
Region	US	AU	US	AU	US	Pacific ex-Japan	Japan	EU ex-UK	UK	North America	EM Latin America	EM Asia	EM Europe & Middle East	US	US	EM	DM	Int	Int	US	US	EM	EAFE	US	US	US	US	US	
Asset class	S&P 500	Equity	S&P 500	S&P ASX 300	S&P 500	Equity								Large cap	Small cap	Equity		Bond	Real Estate	Real Estate	Large cap	Equity		Aggregate bond	General Index	S&P 500	CRSP vw index	Large cap	Small Cap
Skew	-0.66	1.07	-0.12	-0.49	-0.99	-0.53	0.1	-0.54	0.038	-0.44	-0.59	-0.18	0.27	-0.72	-0.56	-0.76	-0.68	0.17	-0.5	-0.87	-0.65	-0.65	-0.53	-0.19	0.05	-0.65	-1.06	-0.38	0.06
Kurtosis (excess)	0.96	0.47	1.54	0.06	2.02	1.69	0.7	1.06	0.18	0.71	1.54	0.17	2.89	1.03	1.08	1.98	1.7	0.58	2.61	8.67	1.42	2.1	1.4	0.74	3.18	0.93	23.92	1.91	3.56
Notes		Average			Quarterly average																						Daily return		

Source: Frontier Advisors

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